AN OPTIMIZATION PROBLEM WITH FREE BOUNDARY GOVERNED BY A DEGENERATE QUASILINEAR OPERATOR.

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ABSTRACT. In this paper we study the existence, regularity and geometric properties of an optimal configuration to a free boundary optimization problem governed by the p-Laplacian.

1. INTRODUCTION

Let D be a bounded domain in $\mathbb{R}^n$ and φ a positive function defined on it (the temperature distribution of the body D). A classical minimization problem in heat conduction asks for the best way of insulating the body D, with a prescribed amount of insulating material in a stationary situation. This situation also models problems in electrostatic, potential flow in fluid mechanics among others.

The mathematical description of this problem is as follows: fixed a number, $\gamma > 0$ (quantity of insulation material), for each domain Ω surrounding D, such that $|\Omega \setminus D| = \gamma$, we consider the potential u associated to the configuration Ω, i.e., the harmonic function in $\Omega \setminus D$, taking boundary data equal to φ on ∂D and 0 on $\partial \Omega$. The flow of heat (quantity to be minimized) corresponding to the configuration Ω, is given by a nonlocal monotone operator given by,

$$J(\Omega) := \int_{\partial D} \Gamma\big(x, u_\mu(x)\big)dS,$$

where μ is the inward normal vector defined on ∂D. The function $\Gamma \colon \partial D \times \mathbb{R} \to \mathbb{R}$ is assumed to be convex and increasing, on u_μ and continuous on x. Important examples are, $\Gamma(t) = t$ (classical heat conduction problem), $\Gamma(t) = t^p$ (optimal configurations in electrostatics), $\Gamma(x, t) = \max\{t, C(x)\}$ (problems is the material sciences).

In [Teix05], the second author studied the problem of minimizing $J(\Omega)$ among all configurations Ω such that, say, $|\Omega \setminus D| = 1$, where $|A|$ is the volume of the set A. This optimization problem with linear heat flux, i.e., $\Gamma(x, t) = t$, was studied in [AAC86] and [ACS88]. Qualitative geometric properties of the free boundary, namely symmetry, uniqueness and full regularity of the free boundary, was explored in [Teix-1].

In this present paper, we turn our attention to this problem when we allow the temperature itself deform the medium. We assume the influence of the temperature distribution on the medium is proportional to the magnitude of its gradient. These considerations lead us to study this optimization problem when temperature distribution is governed by the p-Laplacian. In other words, the variational problem

we are interested in is, for $1 < p < \infty$

$$(1.1) \qquad \text{Minimize} \left\{ \begin{array}{l} J(u) := \int_{\partial D} \Gamma(x, u_\mu)dS \;\Big|\; u \colon D^C \to \mathbb{R}, \; u = \varphi \text{ on } \partial D, \\ \Delta_p u = 0 \text{ in } \{u > 0\} \text{ and } |\{u > 0\}| = 1 \end{array} \right\}$$

Here, $\Delta_p u := \text{div}\big(|\nabla u|^{p-2}\nabla u\big)$. Several new difficulties appear when dealing with the nonlinear operator Δ_p. One of its main difficult lies in the fact that the p-Laplacian is not uniformly elliptic.

In analogy with the linear case for the Laplacian operator, in this paper we shall restrict ourselves to the heat flux given by $\Gamma(x, t) = t^{p-1}$. In our physical considerations, we will assume the body to be insulated has much smaller volume than the quantity of insulation material. This leads us to consider a constant temperature distribution, say, $\varphi \equiv 1$. With these assumptions, problem (1.1) can be reformulated in terms of the following equivalent version of it:

$$(1.2) \qquad \text{Minimize} \left\{ \begin{array}{l} J(u) := \int_{D^C} |\nabla u|^p dx \;\Big|\; u \colon D^C \to \mathbb{R}, \; u = \varphi \text{ on } \partial D, \\ \Delta_p u = 0 \text{ in } \{u > 0\} \text{ and } |\{u > 0\}| = 1 \end{array} \right\}.$$

In problem (1.2), we may assume φ to be a mere continuous and positive function. When $\varphi \equiv 1$ in problem (1.2), such a problem is equivalent to problem (1.1) with $\Gamma(x, t) = t^{p-1}$. In this paper we will only deal with problem (1.2). We hope to turn our attention to problem (1.1) in its full generality in future research.

From the mathematical point of view, our approach is motivated by recent advances on the free boundary regularity theory for minimum problems with a variable domain of integration involving degenerate quasilinear operators. Namely, D. Danielli and A. Petrosyan, in [DP05], have recently extended the celebrated work of H. Alt and L. Caffarelli [AC81], for the p-Laplacian operator. Our regularity results will relay on suitable modifications of the arguments in [DP05]. Furthermore, we shall establish a free boundary condition that will relate our optimization problem with Bernoulli type problems, similar to the ones studied in [HS00], [DPS03] and [DP05].

2. Mathematical fundaments of the physical problem

In this section we shall introduce the main mathematical tools we shall use throughout the whole paper. Throughout the article, $1 < p < \infty$ and $\Delta_p u$ stands for the p-Laplacian operator

$$\Delta_p u := \text{div}\big(|\nabla|^{p-2}\nabla u\big).$$

Let U be a domain in $\mathbb{R}^N$. Let us recall that for any $\xi \in W^{1,p}(U)$, $\Delta_p \xi \in \big[W_0^{1,p}(U)\big]^*$ and

$$\langle \Delta_p \xi, v \rangle = \int_U |\nabla \xi|^{p-2}\nabla \xi \cdot \nabla v dx.$$

Problem (1.2) presents several difficulties from the mathematical point of view. Our strategy will be to study a penalized version of it, which is sort of a weak formulation of the problem. The idea is to grapple with the difficulty of volume constraint, which is very unstable under limits and makes perturbation arguments quite hard.

From now on, we denote by V the following set

$$V := \left\{ u \in W^{1,p}(D^C) \ \Big|\ u = \varphi \text{ on } \partial D \right\}.$$

The penalized problem is stated as follows: Let $\varepsilon > 0$ be fixed. We consider the function

$$f_\varepsilon := \begin{cases} 1 + \dfrac{1}{\varepsilon}(t-1) & \text{if } t \geq 1 \\ 1 + \varepsilon(t-1) & \text{otherwise.} \end{cases}$$

We then define the penalized functional as

$$(2.1) \qquad J_\varepsilon(u) := \int_{D^C} |\nabla u|^p dx + f_\varepsilon\Big(|\{u > 0\}|\Big).$$

For the moment, we shall be interested in the following minimization problem

$$(2.2) \qquad \min_{u \in V} J_\varepsilon(u).$$

For latter use, given a γ-bilipschitz function f we consider the similar penalized problem of (2.2) given by the functional:

$$(2.3) \qquad J_f := \int_{D^C} |\nabla u|^p dx + f\Big(|\{u > 0\}|\Big).$$

3. Properties of solutions of Problem (2.2)

In this section we shall derive existence of a minimizer for the penalized problem as well as some important nondegeneracy conditions, such as optimal regularity and linear growth away from the free boundary. The proof of Theorem 3.1 will be developed throughout this section. At the end, we shall be able to state a representation theorem that will be crucial to study further regularity properties of the free boundary.

Theorem 3.1. *For each $\varepsilon > 0$ fixed, there exists a minimizer $u_\varepsilon \in V$ for the functional J_ε. Furthermore*

(1) *$u_\varepsilon \geq 0$.*

(2) *$\Delta_p u_\varepsilon$ is a nonnegative Radon measure supported on $\partial\{u_\varepsilon > 0\}$. In particular,*

$$\Delta_p u_\varepsilon = 0 \text{ in } \{x \in D^C \ \big|\ u_\varepsilon(x) > 0\}.$$

(3) *Any minimizer u_ε of problem (2.2) is Lipschitz continuous and for any compact $\mathcal{K} \subset D^C$ there exists a constant $K = K(\mathcal{K}, p, n, \varepsilon, D)$ such that $\|u_\varepsilon\|_{Lip(\mathcal{K})} \leq K$.*

(4) *The function u_ε growths linearly way from the free boundary, i.e., for any compact $\mathcal{K} \subset D^C$, there exist positive constants c, C, depending on dimension, $\mathcal{K}$, p, D and ε, such that*

$$c\,dist\big(x, \partial\{u_\varepsilon > 0\}\big) \leq u_\varepsilon(x) \leq C\,dist\big(x, \partial\{u_\varepsilon > 0\}\big), \ \forall x \in \mathcal{K}.$$

(5) *The free boundary is uniformly dense, i.e., for any compact $\mathcal{K} \subset D^C$ fixed, there exist a constant $c = c(\mathcal{K}, p, n, \varepsilon, D)$ with $0 < c < \frac{1}{2}$, such that*

$$c < \frac{|B_r \cap \{u > 0\}|}{|B_r|} \leq 1 - c,$$

for any ball $B_r = B_r(x)$ centered at some point $x \in \partial\{u_\varepsilon > 0\} \cap \mathcal{K}$.

Proof. The existence of a minimizer u_ε for problem (2.2) follows easily from the fact that for any minimizing sequence u_ε^k, we may assume

- $\nabla u_\varepsilon^k \rightharpoonup \nabla u_\varepsilon$ in $L^p(D^C)$
- $u_\varepsilon^k \to u_\varepsilon$ almost everywhere in D^C,

for some $u_\varepsilon \in V$. Thus,

$$\int_{D^C} |\nabla u_\varepsilon|^p dx \leq \liminf_{k \to \infty} \int_{D^C} |\nabla u_\varepsilon^k|^p dx$$
$$\text{and}$$
$$|\{u_\varepsilon > 0\}| \leq \liminf_{k \to \infty} |\{u_\varepsilon^k > 0\}|.$$

Since f_ε is a continuous and increasing function, we obtain

$$J_\varepsilon(u_\varepsilon) \leq \liminf_{k \to \infty} J(u_\varepsilon^k).$$

Clearly, $u_\varepsilon \geq 0$, otherwise, $J_\varepsilon\big((u_\varepsilon)^+\big) < J_\varepsilon\big(u_\varepsilon\big)$.

Observe that if u_ε is a minimizer, then $J(u_\varepsilon) \leq J(u_\varepsilon - \epsilon\eta)$, for every $\epsilon > 0$ and nonnegative $\eta \in C_0^\infty(D^c)$. Since $\{u_\varepsilon > \epsilon\eta\} \subset \{u\varepsilon > 0\}$ and f is increasing, we have that $f\big(\{u_\varepsilon > \epsilon\eta\}\big) \leq f\big(\{u_\varepsilon > 0\}\big)$ and consequently, we have that $\Delta_p u_\varepsilon$ is a nonnegative Radon measure supported on $\partial\{u_\varepsilon > 0\}$.

Now, we explain the main track and the necessary changes in order to obtain items (3), (4) and (5) . We will follow the lines of [DP05] establishing a sequence of lemmas that are analogous of those in this refereed work. Instead of proving these lemmas with all the details, we shall restrict ourselves to enunciate them and sketch their proofs by pointing out the necessary modifications and sending to [DP05] for further details.

Up to the end of this section, we fix some γ-bilipschitz function f. Consider u a minimizer of the problem 2.3 in some ball B. Denote by v the solution of the Dirichlet problem

$$(3.1) \qquad \begin{cases} \Delta_p v &= 0 \quad \text{in } B \\ v &= u \quad \text{on } \partial B. \end{cases}$$

Following the beginning of Section 3 in [DP05] we obtain:

Lemma 3.2. *There exists a constant $C = C(n, p, \gamma)$ such that*

$$(3.2) \qquad \int_B |\nabla(u - v)|^p \leq C|\{u > 0\} \cap B| \text{ for } p \geq 2 \text{ and}$$
$$\int_B |\nabla(u - v)|^p \leq C|\{u > 0\} \cap B|^{\frac{p}{2}} \Big(\int_B |\nabla u|^p\Big)^{1 - \frac{p}{2}} \text{ for } 1 < p \leq 2.$$

Moreover, the constant C goes to zero when γ goes to zero.

The next Lemma is the analogous of Lemma 3.1 in [DP05], with a similar proof.

Lemma 3.3. *Let f be a given γ- Lipschitz function and u be a bounded minimizer of Problem 2.3 in B_1. Then, $u \in C^\alpha$ in $B_{\frac{7}{8}}$ for some $\alpha = \alpha(n, p) \in (0, 1)$ and*

$$\|u\|_{C^\alpha(B_{\frac{7}{8}})} \leq C(n, p, \|u\|_{L^\infty(B_1)}, \gamma).$$

Following Lemma 3.2 in [DP05], we obtain

Lemma 3.4. *Let u be a bounded local minimizer of Problem 2.3 in B_1 with $u(0) = 0$. Then, there exists a constant $C = C(n, p, \gamma)$ such that*

$$\|u\|_{L^\infty(B_{\frac{1}{4}})} \leq C.$$

Proof. Assume, by contradiction, there exists a sequence u_k of bounded local minimizers of Problem 2.3 in B_1 with $u_k(0) = 0$ and $\max_{B_{\frac{1}{4}}} u_k > k$. In this case, we define:

$$d_k(x) = d(x, \partial\{u_k > 0\})$$

and

$$\mathcal{O}_k = \left\{ x \in B_1; d_k(x) \leq \frac{1 - \|x\|}{3} \right\}.$$

Observe that, since $u_k(0) = 0$, $d_k(x) < \|x\|$ for every $x \in B_1$. On the other hand, $(1 - \|x\|)/3 \geq 1/4$ for every $x \in B_{1/4}$. From this inequality we conclude that $B_{1/4} \subset \mathcal{O}_k$. Now, define

$$m_k := \max_{\mathcal{O}_k}(1 - \|x\|)u_k(x) \geq \frac{3}{4} \max_{B_{1/4}} u_k(x) > \frac{3}{4}k.$$

Consider any maximum point $x_k \in \mathcal{O}_k$ of the function $(1 - \|x\|)u_k(x)$ and observe that

$$(3.3) \qquad u_k(x_k) = \frac{m_k}{1 - \|x\|} > \frac{3}{4}k.$$

Denote by y_k any point in $\partial\{u_k > 0\}$ such that $d_k(x_k) = \|y_k - x_k\|$ and define $\delta_k := \|x_k - y_k\|$. Since $x_k \in \mathcal{O}_k$ we have that $\delta_k \leq \frac{(1 - \|x\|)}{3}$. Then, for every $z \in B_{2\delta_k}(y_k)$:

$$\|z\| \leq \|y_k\| + 2\delta_k \leq \|x_k\| + 3\delta_k \leq \|x_k\| + (1 - \|x_k\|) \leq 1.$$

From this inequality, we have that $B_{2\delta_k}(y_k) \subset B_1$. Now, we claim that $B_{\delta_k/2}(y_k) \subset \mathcal{O}_k$. In fact, if $z \in B_{\delta_k/2}(y_k)$ we have that:

$$d_k(z) \leq \frac{\delta_k}{2} \leq \frac{1 - \|y_k\| + \frac{\delta_k}{2}}{3} \leq \frac{(1 - \|z\|)}{3},$$

where the second inequality is a consequence of $\|y_k\| + \delta_k \leq 1$. Moreover, for $z \in B_{\delta/2}(y_k)$

$$(1 - \|z\|) \geq (1 - \|x_k\|) - \|x_k - z\| \geq (1 - \|x_k\|) - \frac{3}{2}\delta_k \geq \frac{(1 - \|x_k\|)}{2}.$$

As a consequence of this inequality,

$$(3.4) \qquad \max_{\overline{B}_{\delta/2}(y_k)} u_k \leq 2u_k(x_k).$$

By the definition of δ_k, we have that $B_{\delta_k}(x_k) \subset \{u_k > 0\}$. Recall from item (2) of Theorem 3.1 that $\Delta_p u_k = 0$ in $B_{\delta_k}(x_k)$. By the Harnack Inequality for p-harmonic functions, we may conclude that there exists a constant $c = c(n, p) > 0$ such that

$$(3.5) \qquad \min_{\overline{B}_{3\delta/4}(x_k)} u_k \geq cu_k(x_k).$$

In particular,

$$(3.6) \qquad \max_{\overline{B}_{\delta/4}(y_k)} u_k \geq cu_k(x_k).$$

Consider the following scaling of u_k

$$(3.7) \qquad w_k(x) = \frac{u_k(y_k + \frac{\delta}{2}x)}{u_k(x_k)}, \text{ for } x \in B_1$$

Observe that since u is a local minimizer of Problem 2.3, w_k is a minimizer of the analogous problem replacing f_ϵ by $f_k = f_\epsilon/u_k(x_k)$. In other words, w_k is a local minimizer of

$$J_k(w) = \int_{B_{3/4}} \|\nabla w\|^p + f_k(|\{w > 0\}|).$$

Now, we denote by v_k the solution of $\Delta_p v_k = 0$ in $B_{3/4}$ and $v_k - w_k \in W_0^{1,p}(B_{3/4})$. Initially we observe that $\|f_k\|_{Lip}$ converges uniformly to zero, as $k \to \infty$. Since w_k is a minimizer of J_k, by Lemma 3.2 we guarantee the existence of a sequence of positive numbers C_k, that converges to zero as $k \to \infty$, such that

$$(3.8) \qquad \int_{B_{3/4}} \|\nabla(w_k - v_k)\|^p \le C_k.$$

By estimates (3.4) and (3.6) we have that

$$\max_{\overline{B}_1} w_k \le 2, \ \max_{\overline{B}_{1/2}} w_k \ge c \text{ and } w_k(0) = 0.$$

Now notice that w_k is uniformly bounded, thus from Lemma 3.3 we conclude w_k and v_k are uniformly C^α in $B_{5/8}$. By compactness, we may assume (passing to a subsequence, if necessary) $w_k \to w_0$ and $v_k \to v_0$ uniformly on $B_{5/8}$. Equation (3.8) implies that $w_0 = v_0 + K$ in $B_{5/8}$. Since $v_k \to v_0$, we have that $\Delta_p v_0 = 0$ and this implies that $\Delta_p w_0 = 0$. By the strong minimum principle, we have that $w_0 = 0$ in $B_{5/8}$, because $w_0 \ge 0$ and $w_0(0) = 0$. It is a contradiction with $\max_{B_{1/2}} w_0 > c > 0$.

This finishes the proof. $\qquad \square$

Now, we show how one can derive Lipschitz continuity of a solution u of Problem 2.3 using Lemma 3.3 and Lemma 3.4:

Proof of Lipschitz continuity of u**:** First, note that it is sufficient to prove that u is Lipschitz continuous at every point y on the boundary of $\{u > 0\}$, since $\Delta_p u = 0$ on the open set $\{u > 0\}$, and therefore u is $C^{1,\alpha}_{loc}$ on $\{u > 0\}$. Take $y \in \partial\{u > 0\}$ and consider $u_r : B_1 \to \mathbb{R}$ defined by:

$$u_r(x) := \frac{1}{r}u(y + rx).$$

Since u is a minimizer of Problem 2.3 in the ball $B_r(y)$, we obtain u_r is a local minimizer of the same problem in the ball B_1, with $u_r(0) = u(y) = 0$. By Lemma 3.4, we have that $\|u_r\|_{L^\infty(B_{1/4})} < C$, where C is a constant depending only on p, γ and n. Thus, by Lemma 3.3 we conclude $u_r \in C^\alpha(B_{1/4})$. Furthermore, for every $r > 0$

$$\|u\|_{C^\alpha(B_{1/4})} \le C = C(n, p, \gamma).$$

This implies that $\|\nabla u\|_\infty \le C$ and thus u is locally C-Lipschitz (see Theorem 4.2.3 in [EG92]). This finish the proof of item (3) of Theorem 3.1. $\qquad \square$

At this moment, the proofs of item (4) and (5) are similar to the proofs of Corollary 4.3 and Theorem 4.4 in [DP05] respectively. The necessary minor modifications are similar to the ones treated in the proof of Lemma 3.4 and therefore we will skip the details. $\qquad \square$

In the same spirit of [AC81], D. Danielli and A. Petrosyan provided in [DP05] a representation theorem for which Theorem 3.1 puts our minimizers u_ε under the hypothesis of. The next Theorem will play an important hole in the investigation of fine regularity properties of the free boundary.

Theorem 3.5. *Let u_ε be a minimizer of problem (2.2). Then*

(1) $\mathcal{H}^{n-1}\big(\mathcal{K} \cap \partial\{u_\varepsilon > 0\}\big) < \infty$ *for every compact set $\mathcal{K} \subset D^C$.*

(2) *There exists a Borel function q_ε such that*

$$\Delta_p u_\varepsilon = q_\varepsilon \mathcal{H}^{n-1} \lfloor \partial\{u_\varepsilon > 0\},$$

that is, for any $\psi \in C_0^\infty(D^C)$, there holds

$$-\int_{D^C} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \psi \, dx = \int_{\{u_\varepsilon > 0\}} \psi q_\varepsilon d\mathcal{H}^{n-1}.$$

(3) *For any compact set $\mathcal{K} \subset D^C$, there exist positive constants c, C such that*

$$c \leq q_\varepsilon \leq C$$

and

$$c r^{n-1} \leq \mathcal{H}^{n-1}\big(B_r(x) \cap \partial\{u_\varepsilon > 0\}\big) \leq C r^{n-1},$$

for every ball $B_x(r) \subset \mathcal{K}$ with $x \in \partial\{u_\varepsilon > 0\}$.

(4) *For $\mathcal{H}^{n-1}$ almost all points in $\partial\{u_\varepsilon > 0\}$, an outward normal $\nu = \nu(x)$ is defined and furthermore*

$$u_\varepsilon(x+y) = q_\varepsilon(x)(y \cdot \nu)^+ + o(y),$$

where $\frac{o(y)}{|y|} \to 0$ as $|y| \to 0$. This allows us to define $q_\varepsilon(x) = (u_\varepsilon)_\nu(x)$ at those points.

(5) $\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\} \setminus \partial_{\text{red}}\{u_\varepsilon > 0\}) = 0.$

4. A geometric-measure Hadamard's variational formula and the free boundary condition

In this section we suggest a geometric-measure version of the well known Hadamard's variational formula (see [G98]) to deduce the free boundary condition of Problem 2.2. Roughly speaking, given two points in the reduced free boundary, say x_1 and x_2, the idea is to make an inward perturbation around x_1, an outward perturbation around x_2 in such a way that we do not disturb very much the original volume and then compare the optimal configuration to the perturbed one in terms of the functional J_ε. Here are the details.

Let $\rho \colon \mathbb{R} \to \mathbb{R}$ be a nonnegative C^∞ function supported in $[0, 1]$, with, say $\int \rho(t) dt = 1$. Hereafter, we shall write $u = u_\varepsilon$ and fix two points x_1 and x_2 in the reduced free boundary $\partial_{\text{red}}\{u > 0\}$. For any $0 < r < \frac{\text{dist}(x_1, x_2)}{100}$, and $\lambda > 0$, we consider the vector field

$$(4.1) \qquad P_r(x) := \begin{cases} x + \lambda r \rho\left(\dfrac{|x - x_1|}{r}\right)\nu(x_1) & x \in B_r(x_1) \\[2mm] x - \lambda r \rho\left(\dfrac{|x - x_2|}{r}\right)\nu(x_2) & x \in B_r(x_2) \\[2mm] x & \text{elsewhere.} \end{cases}$$

If v is any vector in $\mathbb{R}^n$, from direct computation, we see that

$$(4.2) \quad DP_r(x) \cdot v = v + (-1)^{i+1} \left\{ \lambda \rho'\left(\frac{|x - x_i|}{r}\right) \frac{\langle x - x_i, v\rangle}{|x - x_i|} \right\} \nu(x_i) \ \text{ in } B_r(x_i)$$

Notice that, if λ is small enough, P_r is a diffeomorphism that maps $B_r(x_i)$ onto itself. Indeed, if $\lambda \sup_{[0,1]} \rho'(t) < 1$, P_r is a local injective diffeomorphism. Now, if $\lambda \rho(t) \leq 1 - t$, for $0 \leq t \leq 1$,

$$|P_r(x) - x_i| \leq |x - x_i| + \lambda r \rho\left(\frac{|x - x_i|}{r}\right) \leq r,$$

for any $x \in B_r(x_i)$. Finally, notice that $P_r\Big|_{\partial B_r(x_i)} = Id$, therefore P_r has to be onto.

For each $r > 0$ small enough, we will consider the r-perturbed configuration, v_r implicitly defined by

$$(4.3) \qquad\qquad\qquad v_r(P_r(x)) = u(x).$$

The idea is to compare our optimal configuration $\{u > 0\}$ to its perturbation $\{v_r > 0\}$ in terms of the penalized problem 2.2. An important geometric measure information we shall use is the blow-up limit. For any $r > 0$ small enough and $i = 1, 2$, consider the blow-up sequence, $u_r^i \colon B_1(0) \to \mathbb{R}$, given by

$$u_r^i(y) := \frac{1}{r} u\big(x_i + ry\big).$$

From the blow-up analysis, we know, the set $B_1 \cap \{u_r^i > 0\}$ approaches $\{y \in B_1 \mid \langle y, \nu(x_i)\rangle < 0\}$, as $r \to 0$. Let us compute the change on the volume of the perturbation. More specifically, make use of the Change of Variables Theorem, we obtain

$$(4.4)\qquad
\begin{aligned}
\frac{\left|\big(B_r(x_i) \cap \{v_r > 0\}\big)\right|}{r^n} &= \frac{1}{r^n} \int_{B_r(x_i) \cap \{v_r > 0\}} dx \\[2mm]
&= \int_{B_1 \cap \{v_r(x_i + ry) > 0\}} dy \\[2mm]
&= \int_{B_1 \cap \{u_r^i > 0\}} \det\left(DP_r(x_i + ry)\right) dy \\[2mm]
&\longrightarrow \int_{B_1 \cap \{\langle y, \nu(x_i)\rangle < 0\}} 1 + (-1)^{i+1} \lambda \rho'(|y|) \left\langle \frac{y}{|y|}, \nu(x_i) \right\rangle dy,
\end{aligned}$$

as $r \to 0$. Notice there exists a constant $C(\rho)$ so that, for any unit vector $\nu \in \mathbb{S}^{n-1}$ there holds

$$(4.5) \qquad\qquad C(\rho) \equiv \int_{B_1 \cap \{\langle y, \nu\rangle < 0\}} \rho'(|y|) \left\langle \frac{y}{|y|}, \nu \right\rangle dy.$$

A similar computation shows that

$$(4.6) \qquad\qquad \frac{\left|\big(B_r(x_i) \cap \{u > 0\}\big)\right|}{r^n} \longrightarrow \int_{B_1 \cap \{\langle y, \nu(x_i)\rangle < 0\}} dy,$$

as $r \to 0$. Combining (4.4), (4.5) and (4.6), we conclude

$$(4.7) \qquad \frac{\Big(|\{v_r > 0\}|\Big) - \Big(|\{u > 0\}|\Big)}{r^n} \longrightarrow 0,$$

as $r \to 0$. From the Lipschitz continuity of the penalization f_ε, we obtain

$$(4.8) \qquad f_\varepsilon\Big((|\{v_r > 0\}|)\Big) - f_\varepsilon\Big(|(\{u > 0\}|)\Big) \le \frac{1}{\varepsilon}o(r^n).$$

Now we shall turn our attention to the differential of the perturbation on the p-Dirichlet integral. Initially we observe that

$$(4.9) \qquad \begin{aligned} \frac{1}{r^n}\int_{B_r(x_i)} |\nabla u(x)|^p dx &= \int_{B_1} |\nabla u_r^i(y)|^p dy \\ &= \int_{B_1 \cap \{u_r^i > 0\}} |\nabla u_r^i(y)|^p dy, \end{aligned}$$

once $\partial_{\mathrm{red}}\{u_r^i > 0\}$ is smooth. Now, applying twice the Change of Variables Theorem, taking into account that P_r maps $B_r(x_i)$ diffeomorphically onto itself,

$$(4.10)$$
$$\begin{aligned} \frac{1}{r^n}\int\limits_{B_r(x_i)} |\nabla v_r(x)|^p dx &= \frac{1}{r^n}\int\limits_{B_r(x_i)} |DP_r(P_r^{-1}(x))^{-1} \cdot \nabla u(P_r^{-1}(x))|^p dx \\ &= \frac{1}{r^n}\int\limits_{B_r(x_i)} |DP_r(y)^{-1} \cdot \nabla u(y)|^p| \det\big(DP_r(y)\big)|dy \\ &= \int\limits_{B_1 \cap \{u_r^i > 0\}} |DP_r(x_i + rz)^{-1} \cdot \nabla u_r^i(z)|^p| \det\big(DP_r(x_i + rz)\big)|dz. \end{aligned}$$

Now, from (4.2), using the fact that for any matrix A, with $|A| < 1$, we have $\big(Id + A\big)^{-1} = Id + \sum_{i=1}^{\infty}(-1)^i A^i$, we have

$$(4.11) \quad DP_r(x_i + rz)^{-1} \cdot \nabla u_r^i(z) = \nabla u_r^i(z) - (1)^{i+1}\lambda \frac{\rho'(|z|)}{|z|}\langle z, \nabla u_r^i(z)\rangle \nu(x_i) + o(\lambda).$$

On the other hand,

$$(4.12) \qquad |\det\big(DP_r(x_i + rz)\big)| = 1 + (-1)^{i+1}\lambda \frac{\rho'(|z|)}{|z|}\langle z, \nu(x_i)\rangle.$$

Combining (4.9), (4.10), (4.11) and (4.12), we obtain
$$(4.13)$$
$$\begin{aligned} \frac{1}{r^n}\int\limits_{B_r(x_i)} |\nabla v_r(x)|^p - |\nabla u(x)|^p dx &= (-1)^{i+1}\lambda \int\limits_{B_1 \cap \{u_r^i > 0\}} |\nabla u_r^i(z)|^p \frac{\rho'(|z|)}{|z|}\langle z, \nu(x_i)\rangle dz \\ + \quad (-1)^i\lambda \int\limits_{B_1 \cap \{u_r^i > 0\}} &p|\nabla u_r^i(z)|^{p-2}\frac{\rho'(|z|)}{|z|}\langle z, \nabla u_r^i(z)\rangle\langle \nabla u_r^i(z), \nu(x_i)\rangle dz + o(\lambda). \end{aligned}$$

Again, from the blow-up analysis (see [DP05]), for each $\delta > 0$, we know

$$\nabla u_r^i \to q(x_i)\nu(x_i),$$

uniformly in $B_1 \cap \{\langle y, \nu(x_i) \rangle < -\delta\}$. Therefore, by r-uniform Lipschitz continuity of u_r^i, we have

$$(4.14) \qquad \nabla u_r^i \rightarrow -q(x_i)\nu(x_i)\chi_{B_1 \cap \{\langle y, \nu(x_i)\rangle < 0\}},$$

in $L^p(B_1)$. Letting $r \rightarrow 0$ in (4.13), we find

$$(4.15) \qquad \begin{aligned} &\frac{1}{r^n} \int_{B_r(x_i)} |\nabla v_r(x)|^p - |\nabla u(x)|^p dx \longrightarrow \\ &(-1)^{i+1}(p-1)\lambda\big(q(x_i)\big)^p \int_{B_1 \cap \{u_r^i > 0\}} \frac{\rho'(|z|)}{|z|} \langle z, \nu(x_i) \rangle dz + o(\lambda). \end{aligned}$$

Notice that

$$\operatorname{div}\big(\rho(|z|)\big) = \frac{\rho'(|z|)}{|z|} \langle z, \nu(x_i) \rangle,$$

Thus, from Divergence Theorem and the blow-up analysis,

$$(4.16) \qquad \int_{B_1 \cap \{u_r^i > 0\}} \frac{\rho'(|z|)}{|z|} \langle z, \nu(x_i) \rangle dz \rightarrow \int_{B_1 \cap \{\langle z, \nu(x_i)\rangle = 0\}} \rho(|z|)d\mathcal{H}^{n-1}(z) = c(\rho).$$

Putting (4.15) and (4.16) together, we obtain

$$(4.17) \quad \int_{D^C} |\nabla v_r(x)|^p - |\nabla u(x)|^p dx = r^n \lambda(p-1)c(\rho)\big(q(x_1)^p - q(x_2)^p\big) + r^n o(\lambda).$$

From the minimality property of u, (4.8) and (4.17),

$$(4.18) \quad 0 \le J_\varepsilon(v_r) - J_\varepsilon(u) \le r^n \lambda(p-1)c(\rho)\big(q(x_1)^p - q(x_2)^p\big) + r^n o(\lambda) + \frac{1}{\varepsilon}o(r^n)$$

Dividing (4.18) by r^n and letting $r \rightarrow 0$ we obtain

$$(4.19) \qquad 0 \le \lambda(p-1)c(\rho)\big(q(x_1)^p - q(x_2)^p\big) + o(\lambda)$$

Now dividing (4.19) by λ, letting $\lambda \rightarrow 0$, and afterwards reversing the places of x_1 and x_2, we finally obtain

$$(4.20) \qquad\qquad q(x_1) = q(x_2)$$

Since x_1 and x_2 were taking arbitrarily in $\partial_{\mathrm{red}}\{u > 0\}$, we have proven

Theorem 4.1. *There exists a positive constant λ_ε such that*

$$q_\varepsilon \equiv \lambda_\varepsilon, \ \forall x \in \partial_{red}\{u_\varepsilon > 0\}.$$

It now follows from [DP05] that for each $\varepsilon > 0$ fixed, the reduced free boundary $\partial_{\mathrm{red}}\{u_\varepsilon > 0\}$ is a $C^{1,\alpha}$ smooth surface. Real analyticity of the reduced free boundary is then a consequence of [KNS78]. It is worthwhile to point out that, for $n = 2$, a small variant of the main result in [DP] assures full regularity of the free boundary $\{u_\varepsilon > 0\}$, as long as $p > 2 - \sigma$, for some universal constant σ.

In [C87], [C88] and [C89], L. Caffarelli introduced and developed the, by now, well known notion of viscosity solution of a given free boundary problem (for the Laplacian operator). Our final goal of this section is to establish the free boundary condition obtained in Theorem 4.1 in the viscosity sense. This will be used as a geometric tool in the remaining sections.

Theorem 4.2 (Free boundary condition in the viscosity sense). *Let $x_0 \in \partial\{u_\varepsilon > 0\}$ be a free boundary point. Suppose there exists a touching ball B, i.e., $\partial B \cap \partial\{u_\varepsilon > 0\} = \{x_0\}$, such that either $B \subset \{u_\varepsilon > 0\}$ or $B \subset \{u_\varepsilon = 0\}$. Then*

$$(4.21) \qquad u_\varepsilon(x) = \lambda_\varepsilon \langle x - x_o, \nu \rangle^+ + o(|x - x_0|),$$

where λ_ε is the positive constant provided in Theorem 4.1 and ν is the unit normal vector to ∂B, pointing inward to $\{u_\varepsilon > 0\}$.

Proof. Let $B = B_r(\xi)$. We shall first deal with the hypothesis that $B \subset \{u_\varepsilon > 0\}$. For notation convenience, we will omit the center of the ball B. With no loss of generality, we can assume $x_0 = 0$ and $\nu = e_n$. Let Θ be the auxiliary function solving

$$(4.22) \qquad \begin{cases} \mathrm{div}\big(|\nabla u_\varepsilon|^{p-2} D\Theta\big) &= 0 & \text{in } \Delta = B_r \setminus \overline{B_{r/2}} \\ \Theta &= 1 & \text{on } \partial B_{r/2} \\ \Theta &= 0 & \text{on } \partial B_r \end{cases}$$

By the nondegeneracy of the gradient ∇u_ε, see [DP05], $\Theta \in C^{1,\alpha}(\overline{\Delta})$ and since it vanishes on ∂B, we have for some constant $C = C(n, p, \varepsilon, r)$

$$\Theta(x) = C x_n + o(|x|).$$

Consider $\tilde{B}$ to be a ball centered at 0 such that $B \subset\subset \tilde{B}$. We define

$$\theta_0 = \sup\left\{ m : u_\varepsilon(x) \geq m\Theta(x) \text{ in } \tilde{B} \cap B \right\}.$$

and any for $k \geq 1$,

$$\theta_k = \sup\left\{ m : u_\varepsilon(x) \geq m\Theta(x) \text{ in } 2^{-k}\tilde{B} \cap B \right\}.$$

The sequence $\{\theta_k\}_{k \geq 1}$ is increasing and bounded by a constant K_ε, since u_ε is Lipschitz continuous. Let $\tilde{\theta} = \sup_k \theta_k$. If we set

$$\theta := C\tilde{\theta}$$

we have

$$u_\varepsilon(x) \geq \tilde{\theta}\Theta(x) + o(|x|) = \theta x_N + o(|x|).$$

We claim,

$$(4.23) \qquad u_\varepsilon(x) = \theta x_N + o(|x|).$$

Indeed, assume there exists a sequence $(x_k)_{k \geq 1}$ such that $|x_k| = r_k \to 0$ and for some $\delta_0 > 0$,

$$u_\varepsilon(x_k) - \tilde{\theta}\Theta(x_k) > \delta_0 |x_k|.$$

From the definition of θ_k, we can choose a subsequence k_j, where $2^{-6k_j} < r_{k_j} \leq 2^{-4k_j}$ and

$$(4.24) \qquad u_\varepsilon(x_{k_j}) - \tilde{\theta}_{k_j}\Theta(x_{k_j}) > \delta_0 |x_{k_j}|.$$

Now, the function

$$\alpha(x) = u(x) - \theta_{k_j}\Theta(x)$$

satisfies $\mathrm{div}\big(|\nabla u_\varepsilon|^{p-2} D\alpha\big) = 0$ and $\alpha \geq 0$ in $2^{-k_j}\tilde{B} \cap B$. Since u_ε is Lipschitz continuous, from (4.24), we conclude

$$u_\varepsilon(x) - \theta_{k_j} h(x) \geq C\delta_0 |x_{k_j}|$$

in a fixed portion of $\partial B_{r_{k_j}}$. From the Poisson representation formula, we have, in, $\frac{1}{2}\tilde{B}_{r_{k_j}} \cap B$

$$(4.25) \qquad u_\varepsilon(x) - \theta_{k_j}\Theta(x) \geq \overline{C}\delta_0|x_{k_j}|.$$

Now for j large enough, we have

$$u_\varepsilon(x) \geq \left(\theta_{k_j} + \overline{\delta_0}\right)\Theta(x)$$

which is a contradiction with the definition of θ_{k_j}. We have proven so far, $u_\varepsilon(x) = \theta x_N + o(|x|)$. It remains to show, $\theta = \lambda_\varepsilon$. Consider then the blow-up sequence

$$(4.26) \qquad u_\varepsilon^k(x) = \frac{1}{\rho_k}u_\varepsilon(x_0 + \rho_k x),$$

as $\rho_k \to 0$. By Lipschitz continuity and (4.23), we know u_ε^k converges uniform in compact subsets to

$$u_\varepsilon^\infty(x) = \theta\langle x, \nu\rangle^+.$$

However, by a small modification of a classical argument, see [AC81] or [Fr82], we can show the limit of a blow-up sequence of u_ε is an absolute minimizer of J_ε in any ball. Since the free boundary of u_ε^∞ is smooth, we obtain from Theorem 4.1, $\theta = \lambda_\varepsilon$, as desired.

The case $B_r \subset \{u_\varepsilon = 0\}$, follows by a small modification of the above arguments with the fact that Lemma A1 in [C89] also holds for nonnegative Lipschitz functions v satisfying $\text{div}\left(|\nabla u_\varepsilon|^{p-2}Dv\right) \leq 0$. $\qquad\square$

5. Recovering the original problem

In this section we shall relate a solution to the penalized problem to a (possible) solution to our original problem. Roughly speaking the idea is that the function f_ε will charge a lot for those configurations that have a volume bigger than 1. We hope if the charge is too big, i.e., if $\varepsilon > 0$ is small enough, optimal configurations of Problem 2.2 will rather prefer to have volume 1 than paying for the penalization. We will follow the lines of [Teix05].

Lemma 5.1. *There exist positive constants c and C, independent of ε, such that*

$$c \leq |\{u_\varepsilon > 0\}| \leq 1 + C\varepsilon$$

Proof. Let $D^\star$ be any smooth domain containing D, so that $|D^\star \setminus D| = 1$. From the minimality of u_ε, we have

$$(5.1) \qquad J_\varepsilon(u_\varepsilon) = \int_{D^C} |\nabla u_\varepsilon(x)|^p + f_\varepsilon\left(|\{u_\varepsilon > 0\}|\right) \leq J_\varepsilon(u^\star) = C,$$

where $u^\star$ is the p-harmonic function in $D^\star \setminus D$ taking boundary data equal to φ on ∂D and 0 on $\partial D^\star$. Thus

$$\frac{1}{\varepsilon}\left(|\{u_\varepsilon > 0\}| - 1\right) \leq f_\varepsilon\left(|\{u_\varepsilon > 0\}|\right) \leq C.$$

This proves the estimate from above. Let us turn our attention to the estimate by below. Expression (5.1), together with Poincaré inequality, provides

$$(5.2) \qquad \int_{D^C} |\nabla u_\varepsilon(x)|^p + |u_\varepsilon(x)|^p dx \leq C,$$

for some C independent of ε. Let D_δ be a tubular neighborhood of ∂D. For each $x_0 \in \partial D$ fixed, let us consider the fiber $F_{x_0} := \{x_0 + t\mu(x_0) \mid 0 \leq t \leq \delta\}$ and denote

$\Theta_{x_0} := \mathcal{H}^1\big(\{u_\varepsilon > 0\} \cap F_{x_0}\big)$. From Mean Value Inequality, followed by Hölder Inequality and (5.2) we have

$$(5.3) \quad \begin{aligned} \delta\varphi(x_0) &\leq \int_0^\delta u\big(x_0 + t\mu(x_0)\big)dt + \int_0^\delta |\nabla u\big(x_0 + \bar{t}\mu(x_0)\big)|tdt \\ &\leq C\Theta_{x_0}^{1/q}\delta^{1/q}\left(1 + \frac{\delta}{q+1}\right). \end{aligned}$$

Now we integrate (5.3) over ∂D and obtain

$$(5.4) \qquad \int_{\partial D} \varphi dS \leq C(\delta)|\{u_\varepsilon > 0\} \cap D_\delta|^{1/q}.$$

Finally, from (5.4), there must exist a constant, independent of ε, so that $|\{u_\varepsilon > 0\}| \geq c$, as claimed. $\qquad\square$

Lemma 5.2. *There exists a positive constant C independent of ε, so that $\lambda_\varepsilon \leq C$, where λ_ε is the constant provided by Theorem 4.1.*

Proof. Applying Divergence Theorem to the field $F_1 = u|\nabla u|^{p-2}\nabla u$, we have

$$(5.5) \qquad \int_{D^C} |\nabla u_\varepsilon|^p dx = \int_{\partial D} \varphi |\nabla u|^{p-2}\partial_\mu u dS,$$

where μ is the outward unit vector in ∂D. If we apply Divergence Theorem to the field $F_2 = |\nabla u|^{p-2}\nabla u$, we obtain

$$(5.6) \qquad \lambda_\varepsilon^{p-1}\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\}) = \int_{\partial D} |\nabla u|^{p-2}\partial_\mu u dS.$$

Isoperimetric inequality gives a universal bound by below to $\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\})$, i.e, $\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\}) \geq c$, for some c independent of ε. Combining this with (5.5) and (5.6), we obtain

$$\lambda_\varepsilon \leq C(D, \varphi),$$

as claimed. $\qquad\square$

Lemma 5.3. *There exists a universal positive constant $c > 0$, such that $\lambda_\varepsilon \geq c$, for all $\varepsilon > 0$.*

Proof. Let $z_1 \in D^C$ be such that $u_\varepsilon(z_1) > 0$ for all $\varepsilon > 0$. Let us denote by $\delta = \text{dist}(z_1, \partial D)$. Consider the smooth family of domains $\Upsilon_t := B_{\frac{\delta}{2}+t}(z_1) \bigcap D^C$. Let t_ε denote the first t such that Υ_t touches $\partial\{u_\varepsilon > 0\}$. Let us call $x_0 = \partial\Upsilon_{t_\varepsilon} \bigcap \partial\{u_\varepsilon > 0\}$. Define Ψ_ε to be p-harmonic function in $\Upsilon_{t_\varepsilon} \setminus \Upsilon_0$, with the following boundary values data:

$$\Psi_\varepsilon\big|_{\partial\Upsilon_0} = \min_{\partial D} \varphi \quad \text{and} \quad \Psi_\varepsilon\big|_{\partial\Upsilon_{t_\varepsilon}} = 0.$$

By the maximum principle we have $u_\varepsilon \geq \Psi_\varepsilon$ in $\Upsilon_{t_\varepsilon} \setminus \Upsilon_0$. From Hopf's Lemma (see for instance [V84]) we also know there exists a constant $c > 0$ depending on ∂D and $\inf \varphi$, but independent of ε, such that

$$(5.7) \qquad \Psi_{-\nu}(x_0) \geq c,$$

where ν denotes the outward unit normal vector of $B_{\frac{\delta}{2}+t_\varepsilon}(z_1)$, at x_0. Recall from Theorem 4.2, we have the following asymptotic development around x_0

$$(5.8) \qquad \Psi(x) \leq u(x) = \lambda_\varepsilon \langle x - x_0, \nu\rangle^+ + o(|x - x_0|).$$

Dividing (5.8) by $|x - x_0|$, letting $x \to x_0$ and taking into account (5.7), we finally obtain

$$c \le \lambda_\varepsilon,$$

as desired. $\square$

We are ready to show the main theorem of this section.

Theorem 5.4. *If ε is small enough, then any solution of Problem (2.2) is a solution of Problem (1.2).*

Proof. Let us initially suppose $|\{u_\varepsilon > 0\}| > 1$. In the same spirit of Section 4, consider a inward perturbation of the set $\{u_\varepsilon > 0\}$ with volume change V, in such a way that the set of positivity of the new function, $\widetilde{u}_\varepsilon$ is still bigger than 1. Thus

$$(5.9) \qquad f_\varepsilon(|\{\widetilde{u}_\varepsilon > 0\}|) - f_\varepsilon(|\{u_\varepsilon > 0\}|) = -\frac{1}{\varepsilon}V.$$

From (4.13) and Lemma 5.2, we have

$$(5.10) \qquad \int_{D^C} |\nabla \widetilde{u}_\varepsilon|^p - |\nabla u_\varepsilon|^p \quad = \quad \lambda_\varepsilon^p V + o(V)$$
$$\le \quad C^p V + o(V).$$

Using the fact that $J_\varepsilon(u_\varepsilon) \le J_\varepsilon(\widetilde{u}_\varepsilon)$, (5.9) and (5.10), we find

$$(5.11) \qquad 0 \le C^p V + o(V) - \frac{1}{\varepsilon}V.$$

Finally if we divide inequality (5.11) by V and let $V \to 0$, we obtain

$$\varepsilon > \frac{1}{C^p}.$$

If $|\{u_\varepsilon > 0\}| < 1$, we argue similarly, making an outward perturbation and using Lemma 5.3 to obtain another lower bound for ε. Thus, if ε is small enough, $|\{u_\varepsilon > 0\}|$ automatically adjusts to be equal to 1. $\square$

6. Radial Symmetry

In this section we show a simple symmetry result of Problem (1.2). Indeed, we shall show the best way of insulating a uniformly heated spherical body is by a ball. Recall, when $\varphi \equiv$ Constant, Problem (1.2) is equivalent to our original physical optimization problem. Here is the theorem:

Theorem 6.1. *Let D be the unit ball and $\varphi \equiv 1$. Then Problem (1.2) has a unique solution and it is radially symmetric. In particular the free boundary is a sphere.*

Proof. Let $u = u_\varepsilon$ be a solution to Problem (1.2), with $D = B_1$ and $\varphi \equiv 1$. Denote $\Omega = \{u > 0\}$. Let B_{r_1} and B_{r_2} be the biggest ball inside $\Omega \setminus D$ and the smallest ball outside Ω, respectively. Let $y_1 \in \partial B_{r_1} \cap \partial \Omega$ and $y_2 \in \partial B_{r_2} \cap \partial \Omega$. Consider, h_i, $i = 1, 2$ solution of

$$(6.1) \qquad \begin{cases} \Delta_p h_i & = & 0 \text{ in } B_{r_i} \setminus B_1 \\ h_i & = & 1 \text{ on } \partial B_1 \\ h_i & = & 0 \text{ on } \partial B_{r_i}. \end{cases}$$

It is simple to show h_i is radially symmetric. Indeed, h_i is the unique minimizer of

$$E_p(f) := \int_{B_{r_i} \setminus B_1} |\nabla f(x)|^p dx,$$

among all functions $f \in W^{1,p}$ satisfying the according boundary data. For any orthonormal transformation $\mathcal{O} \in O(n)$, consider

$$h_i^{\mathcal{O}}(x) := h_i(\mathcal{O}x).$$

Clearly, $h_i^{\mathcal{O}}$ has the same boundary data as h_i and furthermore,

$$E_p(h_i^{\mathcal{O}}) = \int_{B_{r_i} \setminus B_1} |\mathcal{O}^T \nabla h_i(\mathcal{O}x)|^p dx = E_p(h_i).$$

Thus $h_i^{\mathcal{O}} = h_i$. Since $\mathcal{O} \in O(n)$ was taken arbitrarily, h_i has to be radial.

In particular, the inward normal derivative of h_i over ∂B_{r_i} is a positive constant λ_i. Since $r_1 \leq r_2$, we have

$$(6.2) \qquad\qquad\qquad\qquad \lambda_1 \geq \lambda_2.$$

Now, from the maximum principle,

$$h_1 \leq u \leq h_2.$$

Hence, from the free boundary condition in the viscosity sense, Theorem 4.2, we obtain

$$(6.3) \qquad\qquad \lambda_2 = (h_2)_\nu(y_2) \leq \lambda_\varepsilon \leq (h_1)_\nu(y_1) = \lambda_1.$$

Combining (6.2) and (6.3), we conclude $\lambda_1 = \lambda_2$, and therefore, $r_1 = r_2$. This implies $\partial\Omega$ has to be a sphere of radius $r_1 = r_2$.

We have proven any solution to Problem (1.2), with $D = B_1$ and $\varphi \equiv 1$ is radially symmetric. Uniqueness now follows due to the volume constraint. $\qquad\square$

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